Preface

The Common Core State Standards represent a fundamental change in thinking about school mathematics. In terms of procedural skills, there is little change, and that change in the direction of broadening the uses of those skills. Along with the necessity of developing fluency in basic skills,the most significant difference is that the emphasis is on understanding the actual mathematical content. Algebraic thinking permeates the curriculum from the primary grades, and develops in abstraction through grades 6,7,8. Geometry in 7th and 8th grades is now based on the development of intuition on rigid motions and dilations, as a lead-in to the deeper study in high school. In the elementary grades students learn how to gather and work with data; then in grades 7 and 8, students begin to apply statistical concepts to the analysis of data. Most importantly, emphasis is on deepening student understanding of the central strands of ideas, their flexibility and their applicability, while continuing to require proficiency in technique and procedures.

That emphasis is at the core of the “fundamental change”: from that of viewing mathematics as a collection of tools and protocols for solving problems involving one or more of number, geometry and data, to that of seeing mathematics as a small collection of “strands of knowledge” that develop more or less continuously through the curriculum, providing the user with a source of basic understanding that can be called on and adapted in a wide variety of circumstances.

The first six years of school education are devoted to developing the core of mathematical knowledge that is expected of all functioning adults. The last four years, grades 9-12, develop critical thinking skills in the context of that knowledge and its extensions, in more and more complex situations. Grades 7 and 8 provide the transition from basic knowledge to creative application of that knowledge by consolidating the themes of the previous 6 years in a form that allows an expansion of those ideas, and then beginning the development of that expansion, preparing students for the versatile use to be developed in high school. Success in this transition is critical to the purposes of school education: students must be willing to extend themselves through experimentation and imagination without fear (in high school, in one’s profession, in life); and for that they need a strong foundation, and the experience of moving through unknown terrain with exploratory zeal.

The intent of this text is to introduce tools and guiding ideas to be used by students, teachers and parents as they maneuver this transition; to ground students with mathematical life skills; and to give them the background needed to succeed in high school mathematics and science. These tools are to be made available on a variety of platforms. This was the vision of the Utah State Legislature when they provided funding (through SB 217 2012) for the development of a mathematics text for grades 7 and 8. In 2012, a group consisting of outstanding experienced teachers, teacher educators, and mathematicians proposed a response to this vision. These materials are the result of our work over the subsequent two years, including testing in pilot schools and revision based on their feedback.

The guiding principle for this work is that what happens in the classroom is education, and the teacher is its agent. Success hinges on student engagement and the influence of outside agents (primarily parents). It is the teacher who encourage the engagement of the students, and central to this is a solid and flexible understanding of the material of the course. The goal of these materials is to provide students, teachers and parents with the means to do their part in the process of education.

More specifically we feel that our role is to provide teachers, students and parents with materials that are varied enough so that they can learn from them, adapt them, and use them in a way that makes them comfortable with the authority they have. The controversy over direct instruction vs. inquiry is, in our view, a distraction from the goal of interconnecting subject matter content and pedagogy. The teacher will organize the classroom around learning in the way that is most suitable to the teacher and students. Our role is to provide materials that are adaptable by the teacher whatever the pedagogical approach.

Components of the Text

A. Student Workbook. This is organized into chapters around content objectives, and in each chapter, a sequence of lessons. Each lesson covers classroom activity and homework for a 50 minute class. Sometmes the demands of the material exceed this limitation; when we recognize this, we say so; but some teachers may see different time constraints, and we defer to the teacher to decide how much time to devote to a lesson, how much of it is essential to the demands of the relevant standard. What is important are the proportions dedicated to the various divisions, so that it all fits into a year's work. Within a lesson, the activities for the students are graduated, so that, in working the problems, students can arrive at an understanding of a concept or procedure. In most cases there is an abundancee of problems, providing the teacher with an opportunity to adapt to specific needs.

B. Teacher Edition of the Workbook. Here, the student workbook is annotated with information of value to the teacher (and parent) giving guides through the activities, solutions to the problems, and cautions about what misunderstandings might arise. The intention is that the teacher will read this in advance, and use it to inform the classroom experience.

C. Glossary. It is a principle that to proceed to understanding, we must first agree on the meaning of words we use. Mathematical definitions provide a test for the appropriateness of the use of a term, but do not explain what the term is about. The CCSS suggests that at first students develop an intuitive grasp of the meaning of a term, and as that intuition deepens, the need for precision becomes clear and then the process culminates in the mathematical formulation. In eighth grade, for many terms (proportionality, change, number) we are at the end of that process, but for many, we are in the intuitive stage of developing intuition (function - both in analyzing a functional relation and in synthesizing one from a table of data, geometric transformation, congruence).In this context, we see the point of a *Glossary* as describing how terms are to be understood, indicating the areas of their usefulness Thus our glossary should not be viewed as part of the codification of mathematics; where understanding a term has reached its end, the text will provide the definition.

D. Mathematical Foundations. Here we provide a contextual development of the mathematics, in terms of what has preceded and what is to come, and provide illustrative examples of the problems students are expected to attack and the procedures to deal with them. This material is essential to the understanding of the teacher, and useful to the parent working to help the student grasp the deeper meaning of any problem. Taken as background, it is valuable to the teacher preparing the lesson. For the parent , the mathematical foundations provide examples of problems and solutions while, for the student, it is a text to ground their understanding, and a source from which to obtain it.

Here we find many worked-out examples to illustrate what the students will do in the workbooks, and to illustrate connections with other parts of the curriculum. Throughout this text one finds passages denoted as **Extension**. The intent here is to provide the teacher with tools to answer questions that might come up, but go beyond the standard-based curriculum. Indeed, these extensions provide tools to raise questions, whose exploration leads the student to a richer understanding of the material (while in many cases previewing what is to come in future years). In this sense, these passages are also designed to maintain and promote the interest of those students who grasp the fundamental ideas and techniques quickly.

In summary, in 7th and 8th grade, the curriculum must affect a transition from the acquisition and consolidation of fundamental mathematical concepts and procedures, to an ability and flexibility in their use to solve problems. There are several critical parts to this transition:

a. Number sense: from counting to measuring;

b. Algebraic thinking: from unknown to variable, from equation to relation to function;

c. Geometry: from a study and classification of shape and size to a study of attributes of geometric figures that remain the same under transformation;

Through these areas an important transition is begun: from *static* to *dynamic* - from understanding how things are to understanding how they change, We see this in:

- Number: from the sense of counting objects, or parts of objects, to moving along a line in tenths and hundredths and so on to assign a numerical expression to a point;
- Algebra: from the idea that we have expressed relations among specific numbers, some of which are unknown to the idea that we are expressing a relation between two variables, and that we sweep along explicit values of this relation to discern a pattern of change
- Geometry: from the study of the shape of figures to the comparison of figures and ways to move one figure onto the other, if possible.

This is a tall order; too much to be accomplished in two grades. But the Core Standards account for this and do not ask for *accomplishment*, but an expansion of thinking prerequisite to accomplishment. Much has happened in prior grades to prepare students for these transitions, and the goal by the end of eighth grade is that students are eager to make them.

These materials were created over a two year period, 2012-2014, tested at pilot schools in Utah during the academic year 2013-2014 with a final revision completed in Fall, 2014. Fundamental to this work was a functional collaboration among teachers, mathematics educators and mathematicians. Assumptions, priorities and techniques ranged widely from group to group and person to person. We were, however, united in our conviction that the new Core Standards presented mathematics in a functionally different way that is a vast improvement over the past. This conviction led us (sometimes over months) to agreement on the materials; often that agreement was that there are indeed several ways to proceed.

Structure of the Text: Eighth Grade

The text is organized around the central concepts of eighth grade mathematics: Chapters 1 through 4 comprise a unit on *linearity*; Chapters 5 and 6 develop the concept of *function*; Chapters 7 and 8 are on *number* and Chapters 9 and 10 on *geometry*. There is much overlap: linearity plays a central role in the development of function, and the approach to number in Chapters 7 and 8 is strongly based on geometry. Finally, while Chapter 6 covers the statistics of 8th grade, it is about the search for functional relations in bivariate data.

Another way to look at the structure of this text is that the first half is about algebra: equations and functions, then there is a chapter on statistics, and the rest is about geometry. It is good to keep this organization in mind, for it orients the text in an important way; but it should not be taken as the basic structure of the course. In the first half, references to geometry are essential to understanding; and at the same time, algebra is deeply embedded in the statistics and geometry of the second half.

Let us now briefly scan the conceptual high points of the text chapter by chapter.

Chapters 1 through 4: Linear Equations: Representations and Solutions

These chapters constitute a gradual, but deliberate, transformation from statics to dynamics. From asking, “Given certain conditions on the *unknowns* x and y, what are the numbers that they represent?” to “What do these conditions tell us about the relationship between the *variables* x and y?" First students are concerned with solving equations for specific unknowns. The equations are linear; students have solved such equations before, here they develop a protocol to solve the general equation. Gradually this language, of equations and unknowns, is transformed into a language about relations and variables.

Chapter 1. A linear expression is an algebraic formula that can be reduced to the form $ax+b$ using the operations of algebra. In seventh grade, students begin by solving equations where an expression is set equal to a number; so long as the coefficient of the unknown is not zero, there is a unique solution. Then students solve for the situation where two expressions are equated. Now there may be no solution, one solution, or all numbers solve the equation (for linear expressions this is an *equivalence*). The final section of the chapter deals with real-life problems, where the issue is that of formulating information as algebraic equations in order to solve.

Chapter 2. We begin with review of the discussion of proportional relationships in grade 7. There students saw that if y and x are in a proportional relationship, then there is a number r, called the *constant of proportionality* such that $y = rx$. Now we focus on r as a rate, in particular the *unit rate* of y with respect to x: one unit of x corresponds to r units of y. Then, through examples, we move into linear relationships, leading to equations of the form $y = mx + b$, noticing that these are like proportional relationships, in that they are represented graphically by straight lines, but that the line now may not go through the origin. For $x = 0$, y now has the *initial value* b which could be different from 0. The concept of unit rate now is replaced by that of constant *rate of change*, given by m. In the next section, we turn to the study of the graph of a linear relation, and show that m is the rise/run (or {change in y}/{change in x}), called the *slope* of the line. To complete the discussion, we introduce some geometric concepts (translation and dilation) in order to show that a (non-vertical) line is always the graph of a linear relation. The important ingredient is that the quotient {change in y}/{change in x} is always the same number (m), no mather what two points on the line we take.

Chapter 3. This chapter concludes the discussion of linear relations by developing fluidity in moving among the various representations of a linear relation: a) the equation $Ax + By = C$, b) a table of values c) the graph, d) the equation $y = mx + b$. Here we introduce the word *function* by noticing that the equation $y = mx + b$ gives a set of rules to calculate the value of y for any given value of x, and thus establishes that we can consider y as determined by$x$ (that is, y is a function of x.)

Chapter 4. This chapter is a slight diversion from the progression from unknowns and equations to functions and variables: the protocols for finding the values of two unknowns that make two linear equations true simultaneously. Even though this problem is about "unknowns" and not "variables," the understanding of the various cases that can occur (no solutions, one solution, a line of solutions) requires a graphical interpretation in terms of the unknowns in the two equations as variables. Once this occurs, we turn to two procedures for the solution of a pair of simultaneous equations. If one of the equations expresses one of the unknowns in terms of the other (as in $x = ay + b$), then the quickest way is to *substitute* $ay + b$ for x in the other equation, and solve for y. The second method, *elimination*, works for any pair of simultaneous equations, and in fact is the method that generalizes to systems of many equations in many unknowns (looking forward to secondary mathematics).

Chapters 5 and 6: Functions: Analyzing and Synthesizing

Chapter 5. The purpose of this chapter is to develop an intuitive understanding of the concept of *function*, and its representation, through examples. Now it is essential that the student stop thinking of x and y as "unknowns to be found," but as a range of pairs of numbers (x, y) to be traversed, subject to possible conditions. We may sample the range, and in that way develop a table of corresponding values. In Chapter 5, the range is given by the relation; in Chapter 6, we start with samples and try to induce from that information what the field looks like.

But, what we have been calling the range is the important abstraction: an infinite set of pairs that are in the relationship. If the conditions tell us that to any x, there corresponds only one y such that (x, y) is in the field, then we say that y is a *function* of x. Ordinarily, and especially in eighth grade, the conditions for y to be a function of x consist of a specific set of instructions telling us how to find the y corresponding to an x. Although this chapter is filled with examples, the interest is not in how to represent the rules for the function, but how to understand what they tell us about the relation of the variables x and y (or r and A), or however we label the quantities being measured.

The idea of representing a number by a letter is a major step in the progression into algebra: by seventh grade students should be used to this idea. But then in eighth grade, we jump to the use of letters to represent a range of numbers (thus the word *variable*). This jump doesn't come easy, and students should be prepared for it. But then comes a deeper plunge into abstraction: using a letter (f) to represent a set of rules (the *function*). It will take time for students to adjust to and accept this ever-increasingly abstract use of letters; more time than one week or one course. In 8th grade the process is begun by helping students to become accustomed to these abstractions, specifically in the context of linear functions; stressing that the characteristic is that of constant rate of change.

Chapter 6. In Chapter 5 we began the discussion of the concept of *function* as a set of rules that instruct us on how to find, given a value of one of the variables (usually x), the corresponding value of the other variable (usually y). This is a long story, reaching well into secondary mathematics. For scientists studying the possibility of a causal relation between two variables, the first problem is to find the rule, and then to test its implementation. This is the domain of *bivariate statistics* to which Chapter 6 is an introduction. If the rule is shown to be quite accurate (in its implementation) at predicting values of y from values of x, this is called a *strong correlation* but the scientist still has the problem to explain why there is this relation. In studying bivariate statistics, it is important to distinguish between correlation and causality: the statistical study determines whether or not there is a correlation, and how good it is; but the reason for the correlation remains in the realm of the science.

Statistical analysis provides formulae to give us a *best fitting line* for the data. These formulae are based on a concept of distance between data sets. Given a bivariate data set as a collection of points on a plane, we can find a line that, among all data sets lying on a line, is the closest to the given data. There is also a measure of "how close" given by the *covariance* of the data. These formulae are not part of the 8th grade curriculum, because, at this point, an intuitive understanding of what is going on is more important. Does it make sense to look for a best fitting line to model the data? If so, can we eyeball it? The interesting fact is that, if the graph of the data set follows, in some sense, a linear pattern, we can eyeball the linear model (the best fitting line) quite closely. And, equally important: if the graph of the data fails to show a linear tendency, then we better abandon the search for one.

It is a common misunderstanding that science is a way to collect and organize facts, and if the number of facts is large, and the organization logical, we have a theory. In reality, science starts with an intuition, creates tests of that intuition, and modifies, rejects, or moves forward with that intuition, depending upon the results of the tests. When the intuition (now substantially modified and developed in detail as a result of tests) provides a basis that logically connects the results of the experiments, and is always reinforced by the results of further tests, then it becomes a theory: one that can direct the building of skyscrapers or the research of potential medicines, but one that still does not embody absolute truth.

Chapters 7 and 8: Numbers and Measures, Large and Small

Chapter 7. In Chapter 7 we return to the number sense strand. The context is that of the number line; we at first review the placing of numbers on the number line. Then, in section two, we raise the question: what is the number corresponding to a given length? This is done in a concrete context, that of the hypotenuse of a right triangle whose sidle lengths are integers. The strategy of tilted square (a brilliant idea based on one of the proofs of the Pythagorean Theorem) gives students the opportunity (by calculating and comparing areas) to find squares of area 5, 9, 13. So, then, what is the measure of the length of a side of the square? In attempting to answer that question, students begin to see that it is not easy to assign a quotient of integers to the length of the side of a square of area 5,9,13, unless, as is the case for 9, the number is the square of an integer. In fact, through more examples, students come to see that, for any positive integer, its square root is either an integer or is irrational.

In section 3, we recall discussions of decimal expansions from seventh grade, in particular that for a rational number, its decimal expansion is either terminating or eventually repeating. Here we develop techniques to show the converse: that a terminating and eventually repeating decimal is the decimal expansion of a rational number.

One of the strange anomalies is that students (middle school, high school, college) readily accept the idea that a non-repeating decimal identifies a point on the line that is not a rational number, even though this is a very abstract concept. But they reel at the idea that they can, with only straight edge and compass, create a square of area 2 units, but cannot realize the length of the side of the square as a quotient of integers. Well, maybe not a strange anomaly: 2500 years ago, the Pythagoreans struggled with the same discovery.

Chapter 8. This chapter is about numbers: how to represent them and how these new notations make their use more flexible. The first section, about integer exponents, recalls the use of A^2 and A^3 to represent the multiplication of two As and then three As. Then we extend that notation to any positive integer as exponent exponents: A^n represents the multiplication of n As together. We observe that the laws of arithmetic applies to exponents:

$$A^{n+m} = A^n \times A^m , \quad A^{nm} = (A^n)^m .$$

A^0 need some explanation, but in 8th grade, we can justify the definition $A^0 = 1$ (for nonzero A) by imposing the laws of arithmetic. For the second formula above to continue to hold, we must have $(A^0)^2 = A^{0\times 2} = A^0$, and the only number (other than 0) equal to its square is 1. What about 0^0? Discussion of this question can lead to interesting conjectures, but the end result should be that it is *not definable*.

The rules of arithmetic, allow us to extend exponential notation to negative integers. Again to illustrate: for the rules of arithmetic to continue to hold, A^{-4} has to satisfy the requirement that $A^4 \times A^{-4} = A^{4+(-4)} = A^0 = 1$. Thus A^{-4} is the multiplicative inverse of A^4. Replacing 4 with any number, we get the rule

$$A^{-n} = \frac{1}{A^n} .$$

In the next section, we exploit exponential notation to make useful comparisons between very large numbers or between very small numbers. The term *scientific notation* is given to to any use of the notation $a \times 10^n$, where a any number, called the *significant figure*, and n is an integer, called the *order of magnitude*. The notation is said to be *normalized* if the integral part of a is between 1 and 9. It helps in understanding scientific notation to compare it to place value, as in noting that 10^{17} is a one followed by 17 zeroes.

The last section is about *volume*. In part it is a way to apply the ideas of the preceding sections; but the main purpose is to extend the ideas and techniques of 7th grade (on polyhedral regions) to more general regions (where one or more of the edges of the figure is circular). The fundamental concept here is twofold: first: the volume of a rectangular prism of side lengths a, b, c is $V = abc$, and second, the volume of any object is the sum of the volumes of the rectangular prisms that fill it up. This definition is not quite precise: it may not be possible to fill the figure up with rectangular prisms, in which case the calculation produces an approximation to the volume of the object, which gets better and better as we fill it with smaller and smaller rectangular prisms,

This is clearly the generalization to three dimensions of the concept of area in two dimensions. This leads to Cavalieri's principle, which roughly states: for an object in three dimensions that is formed by drawing a planar figure out into space for a distance h, then its volume is the product of h and the area of the planar figure.

These concepts are used to lead the students to an intuitive understanding of the volume formulas for the cylinder, cone and sphere. At this stage of development, these formulas are not *proven*; instead, the intent is that the students will know them and use them. In 8th grade the emphasis is on giving evidence (both empirical and analogical) for these formulas. The cylinder is easy, since it is a solid formed by drawing a disc out into space in a perpendicular direction. For the cone and sphere, the formulas are justified by relating them to the formula for the cylinder.

Chapters 9 and 10: Explorations into Transformational Geometry

Chapter 9. This chapter is an exploratory introduction to transformational geometry. The fundamental *rigid* transformations (translation, rotation, reflection) are introduced in an operational way: students are asked to perform these transformations using coordinate grids on transparent paper (*patty paper*), and to experiment with them in specific contexts. The first section brings the students through a variety of illustrations of these motions, so that they will understand these transformations as a codification of the idea of moving figures from one position to a different position.

As a result, we are led to the concept of *congruence*: two figures are called *congruent* if there is a sequence of rigid motions taking one to the other.

The central concerns coming from these definitions are:

a. Under what conditions on the size and shape of the figure can we conclude that two figures are congruent?
b. If two figures are congruent what do they have in common?
c. Given two figures that appear to be congruent, find the sequence of motions that can resolve the question.

This issue is explored in a general context, but primarily that of rectangles and triangles. Under what circumstances are two rectangles congruent? Under what circumstances are two triangles congruent? Through exploration, students will find that, for polygons triangles, corresponding sides have the same length (for triangles this is the SSS criterion). They may also discover that this criterion is also sufficient for congruence. For triangles, further criteria for congruence are explored.

This word *exploration* is fundamental to the understanding of the 8th grade curriculum in geometry. Students are to acquire an experimental understanding of congruence in 8th grade, and in Secondary 1, a logical development will be developed. The distinction between proof and experience is subtle, and 8th grade geometry is a lead-in to this discussion, which develops further in Secondary I and II. The Appendix to this text provides a quick survey of a logical development of transformational geometry, so that teachers can see where the subject is heading.

In section 9.2 the geometric issues are further refined: in what way is “of the same shape and size” differ from “of the same shape?” The discussion takes off from the 7th grade discussion of scale drawings, where ”same shape” amounts to a change of units of measurement. In 8th grade, this is made more precise by the concept of *dilation*. Students explore with dilations and through observation discover it s properties. From there, the work goes on to discover conditions for similarity (as in: if corresponding angles of a triangle are congruent, the the triangles are similar).

Chapter 10. Here we concentrate on geometric facts dealing with the measure of angles and line segments. In the first section, we concentrate on angles and triangles. In 7th grade students, through experimentation, came to see that the sum of the angles of a triangle is a straight angle (that is, an angle whose measure is 180°). Here we seek a deeper understanding of this fact, and solve more complex problems using it, and explore other theorems about angles formed by intersecting lines (such as: two lines are parallel if a line intersects both given lines so that corresponding angles are of equal measure).

Section 2 is devoted to the standard 8.G.6: *Explain a proof of the Pythagorean GTheorem and its converse*. Keeping in mind that much of 8th grade is prelude to high school mathematics, we look at the relevant high school standard and find *Prove the Pythagorean Theorem using similar triangles*. Clearly this is not the proof in mind for the 8th grade standard, for students have yet to develop the required facility with quadratic expressions. In this text we take the proofs to be those that are visual, involving two different decompositions of a square related to the theorem.

First, we revisit the tilted square discussion of Chapter 7 and use it as one demonstration of the Pythagorean Theorem (this is hinted at in the Chapter 7 discussion). We then present another visual realization of the Pythagorean Theorem that starts with a square whose side length is that of the hypotenuse.

The converse of the Pythagorean Theorem says this: given a triangle with sides of length a, b, c that satisfy the equation $a^2 + b^2 = c^2$, then the angle opposite the side of length c is a right angle. Using a dynamic model, students see that of all triangles with two side lengths a and b, there is precisely one triangle whose third side is of length $\sqrt{a^2+b^2}$. Since the right triangle with those side lengths has this property, it is the only one with that property.

The last two sections deal with applications of the Pythagorean theorem and its converse, in solving problems involving right triangles, and as a way to find the distance between two points in the coordinate plane, given their coordinates. This leads to the "distance formula" in coordinate geometry. Here again, students should focus on the method for finding the distance between two points, rather than its expression as a formula. The formula is soon forgotten, but the process will remain with the student and support understanding in later grades, particularly in the vector approach to geometry.

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